GABOR FRAMES AND TOTALLY POSITIVE FUNCTIONS

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Abstract. Let g be a totally positive function of finite type, i.e., $\hat{g}(\xi) = \prod_{\nu=1}^{M}(1+2\pi i\delta_\nu\xi)^{-1}$ for $\delta_\nu \in \mathbb{R}$ and $M \geq 2$. Then the set $\{e^{2\pi i\beta lt}g(t-\alpha k) : k,l \in \mathbb{Z}\}$ is a frame for $L^2(\mathbb{R})$, if and only if $\alpha\beta < 1$. This result is a first positive contribution to a conjecture of I. Daubechies from 1990. So far the complete characterization of lattice parameters α, β that generate a frame has been known for only six window functions g. Our main result now provides an uncountable class of functions. As a byproduct of the proof method we derive new sampling theorems in shift-invariant spaces and obtain the correct Nyquist rate.

1. Introduction

The fundamental problem of Gabor analysis is to determine triples (g, α, β) consisting of an L^2-function g and lattice parameters $\alpha, \beta > 0$, such that the set of functions $\mathcal{G}(g,\alpha,\beta) = \{e^{2\pi i\beta lt}g(t-\alpha k) : k,l \in \mathbb{Z}\}$ constitutes a frame for $L^2(\mathbb{R})$. Thus the fundamental problem is to determine the set (the *frame set*)

$$\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \mathcal{G}(g,\alpha,\beta) \text{ is a frame } \} . \tag{1}$$

It is stunning how little is known about the nature of the set $\mathcal{F}(g)$, even after twenty years of Gabor analysis. The famous Janssen tie [20] shows that the set $\mathcal{F}(g)$ can be arbitrarily complicated, even for a "simple" function such as the characteristic function $g = \chi_I$ of an interval.

Under mild conditions, precisely, if g is in the Feichtinger algebra M^1, then the set $\mathcal{F}(g)$ is open in $\mathbb{R}_+^2$ [13]. Furthermore, if $g \in M^1$, then $\mathcal{F}(g)$ contains a neighborhood U of 0 in $\mathbb{R}_+^2$. Much effort has been spent to improve the analytic estimates and make this neighborhood as large as possible [5,9,26]. The fundamental density theorem asserts that $\mathcal{F}(g)$ is always a subset of $\{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta \leq 1\}$ [10,14,17]. If $g \in M^1$, then a subtle version of the uncertainty principle, the so-called Balian-Low theorem, states that $\mathcal{F}(g) \subseteq \{(\alpha,\beta) : \alpha\beta < 1\}$ [4, 8]. This means that $\{(\alpha,\beta) : \alpha\beta \leq 1\}$ is the maximal set that can occur as a frame set $\mathcal{F}(g)$.

Until now, the catalogue of windows g for which $\mathcal{F}(g)$ is completely known, consists of the following functions: if g is either the Gaussian $g(t) = e^{-\pi t^2}$, the hyperbolic secant $g(t) = (e^t + e^{-t})^{-1}$, the exponential function $e^{-|t|}$, then $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$; if g is the one-sided exponential function $g(t) = e^{-t}\chi_{\mathbb{R}^+}(t)$, then $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta \leq 1\}$. In addition, the dilates of these functions

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and their Fourier transforms, $g(t) = (1 + 2\pi i t)^{-1}$ and $g(t) = (1 + 4\pi^2 t^2)^{-1}$, also have the same frame set. The case of the Gaussian was solved independently by Lyubarski [25] and Seip [31] in 1990 with methods from complex analysis in response to a conjecture by Daubechies and Grossman [11]; the case of the hyperbolic secant can be reduced to the Gaussian with a trick of Janssen and Strohmer [22], the case of the exponential functions is due to Janssen [19,21]. We note that in all these cases the necessary density condition $\alpha\beta < 1$ (or $\alpha\beta \le 1$) is also sufficient for $\mathcal{G}(g, \alpha, \beta)$ to generate a frame.

The example of the Gaussian lead Daubechies to conjecture that $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$ whenever g is a positive function in L^1 with positive Fourier transform in L^1 [9, p. 981]. This conjecture was disproved in [18].

Surprisingly, no alternatives to Daubechies' conjecture have been formulated so far. In this paper we deal with a modification of Daubechies' conjecture and prove that the frame set of a large class of functions is indeed the maximal set $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$.

This breakthrough is possible by combining ideas from Gabor analysis, approximation theory and spline theory, and sampling theory. The main observation is that all functions above — the Gaussian, the hyperbolic secant, and the exponential functions — are *totally positive functions*. This means that for every two sets of increasing real numbers $x_1 < x_2 < \cdots < x_N$ and $y_1 < y_2 < \cdots < y_N$, $N \in \mathbb{N}$, the determinant of the matrix $[g(x_j - y_k)]_{1 \le j,k \le N}$ is non-negative.

Indeed, for a large class of totally positive functions to be defined in (15) we will determine the set $\mathcal{F}(g)$ completely. Our main result is the following:

Theorem 1. *Assume that $g \in L^2(\mathbb{R})$ is a totally positive function of finite type ≥ 2. Then $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. In other words, $\mathcal{G}(g, \alpha, \beta)$ is a frame, if and only if $\alpha\beta < 1$.*

This theorem increases the number of functions with known frame set from six to uncountable. We will see later that the totally positive functions of finite type can be parametrized by a countable number of real parameters, see (15). Among the examples of totally positive functions of finite type are the two-sided exponential $e^{-|t|}$ (already known), the truncated power functions $g(t) = e^{-t} t^r \chi_{\mathbb{R}_+}$ for $r \in \mathbb{N}$, the function $g(t) = (e^{-at} - e^{-bt})\chi_{\mathbb{R}^+}(t)$ for $a, b > 0$, or the asymmetric exponential $g(t) = e^{at}\chi_{\mathbb{R}^+}(-t) + e^{-bt}\chi_{\mathbb{R}^+}(t)$, and the convolutions of totally positive functions of finite type. In addition the class of g such that $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$ is invariant with respect to dilation, time-frequency shifts, and the Fourier transform. Since $\mathcal{G}(g, \alpha, \beta) = \mathcal{G}(\hat{g}, \beta, \alpha)$, we obtain a complete description of the frame set of the Fourier transforms of totally positive functions. For instance, if $g(t) = (1 + 4\pi^2 t^2)^{-n}$ for $n \in \mathbb{N}$, then $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$.

To compare with Daubechies' original conjecture, we note that every totally positive and even function possesses a positive Fourier transform. Theorem 1 yields a large class of functions for which Daubechies' conjecture is indeed true. Furthermore, Theorem 1 suggests the modified conjecture that the frame set of every continuous totally positive function is $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$.

Our main tool is a generalization of the total positivity to infinite matrices. We will show that an infinite matrix of the form $[g(x_j - y_k)]_{j,k\in\mathbb{Z}}$ possesses a left-inverse, when g is totally positive and some natural conditions hold for the sequences (x_j) and (y_k) (Theorem 8).

The analysis of Gabor frames and the ideas developed in the proof of Theorem 1 lead to a surprising progress on another open problem, namely (nonuniform) sampling in shift-invariant spaces. Fix a generator $g \in L^2(\mathbb{R})$, a step-size $h > 0$, and consider the subspace of $L^2(\mathbb{R})$ defined by

$$V_h(g) = \{f \in L^2(\mathbb{R}) : f = \sum_{k\in\mathbb{Z}} c_k g(. - kh)\} .$$

For the case $h = 1$ we write $V(g)$, for short. We assume that the translates $g(.-k)$, $k \in \mathbb{Z}$, form a Riesz basis for $V(g)$ so that $\|f\|_2 \asymp \|c\|_2$. Shift-invariant spaces are used as an attractive substitute of bandlimited functions in signal processing to model "almost" bandlimited functions. See the survey [3] for an introduction to sampling in shift-invariant spaces. An important problem that is related to the analog-digital conversion in signal processing is the derivation of sampling theorems for the space $V(g)$. We say that a set of sampling points x_j, ordered linearly as $x_j < x_{j+1}$, is a set of sampling for $V_h(g)$, if there exist constants $A, B > 0$, such that

$$A\|f\|_2^2 \leq \sum_{j\in\mathbb{Z}} |f(x_j)|^2 \leq B\|f\|_2^2 \qquad \text{for all } f \in V_h(g) .$$

As in the case of Gabor frames there are many qualitative sampling theorems for shift-invariant spaces. Typical results require high oversampling rates. They state that there exists a $\delta > 0$ depending on g such that every set with maximum gap $\sup_j(x_{j+1} - x_j) = \delta$ is a set of sampling for $V(g)$ [1, 3, 34]. In most cases δ is either not specified or too small to be of practical use. The expected result is that for $V(g)$ there exists a Nyquist rate and that $\delta < 1$ is sufficient. And yet, the only generators for which the sharp result is known are the B-splines $b_n = \chi_{[0,1]} * \cdots * \chi_{[0,1]}$ ($n + 1$-times). If the maximum gap $\sup_j(x_{j+1} - x_j) = \delta$ satisfies $\delta < 1$, then $\{x_j\}$ is a set of sampling for $V(b_n)$ [2]. (This optimal result can also be proved for a generalization of splines, the so-called "ripplets" [6]). It has been an open problem to identify further classes of shift-invariant spaces for which the optimal sampling results hold.

Here we will prove a similar result for totally positive generators.

Theorem 2. *Let g be a totally positive function of finite type and $\mathcal{X} = \{x_j\}$ be a set with maximum gap $\sup_j(x_{j+1} - x_j) = \delta$. If $\delta < 1$, then $\mathcal{X}$ is a set of sampling for $V(g)$.*

The theorem will be a corollary of a much more general sampling theorem.

The paper is organized as follows: In Section 2 we discuss the tool box for the Gabor frame problem. We will review some known characterizations of Gabor frames and derive some new criteria that are more suitable for our purpose. We then recall the main statements about totally positive functions and prove the main technical theorem about the existence of a left-inverse of the pre-Gramian matrix.

In Section 3 we study Gabor frames and discuss some open problems that are raised by our new results. In Section 4 we prove the sampling theorem.

2. Tools

2.1. **Characterizations of Gabor Frames.** There are many results about the structure of Gabor frames and numerous characterizations of Gabor frames. In principle, one has to check that one of the equivalent conditions for a set $\mathcal{G}(g,\alpha,\beta)$ to be a frame is satisfied. This task is almost always difficult because it amounts to proving the invertibility of an operator or a family of operators on an infinite-dimensional Hilbert space.

In the following we summarize the most important characterizations of Gabor frames. These are valid in arbitrary dimension d and for rectangular lattices $\alpha\mathbb{Z}^d\times\beta\mathbb{Z}^d$. We will use the notation $M_\xi f = e^{2\pi i\xi\cdot}f$ and $T_y f = f(\cdot - y)$, $\xi, t\in\mathbb{R}^d$, such that

$$\mathcal{G}(g,\alpha,\beta) = \{M_{l\beta}T_{k\alpha}g : k,l\in\mathbb{Z}^d\}\,.$$

Then $\mathcal{G}(g,\alpha,\beta)$ is a frame of $L^2(\mathbb{R}^d)$, if there exist constants $A,B>0$, such that

$$A\|f\|_2^2 \le \sum_{k,l\in\mathbb{Z}} |\langle f, M_{l\beta}T_{k\alpha}g\rangle|^2 \le B\|f\|_2^2 \qquad \text{for all } f\in L^2(\mathbb{R}^d)\,.$$

If only the right-hand inequality is satisfied, then $\mathcal{G}(g,\alpha,\beta)$ is called a Bessel sequence.

Following the fundamental work of Ron and Shen [26] on shift-invariant systems and Gabor frames, we define two families of infinite matrices associated to a given window function $g\in L^2(\mathbb{R}^d)$ and two lattice parameters $\alpha,\beta>0$.

The pre-Gramian matrix $P=P(x)$ is defined by the entries

$$P(x)_{jk} = g(x+j\alpha-\tfrac{k}{\beta}), \qquad j,k\in\mathbb{Z}^d\,. \tag{2}$$

The Ron-Shen matrix is $G(x)=P(x)^*P(x)$ with the entries

$$G(x)_{kl} = \sum_{j\in\mathbb{Z}^d} g(x+j\alpha-\tfrac{l}{\beta})\,\bar{g}(x+j\alpha-\tfrac{k}{\beta}), \qquad k,l\in\mathbb{Z}^d\,. \tag{3}$$

Theorem 3 (Characterizations of Gabor frames)**.** *Let $g\in L^2(\mathbb{R}^d)$ and $\alpha,\beta>0$. Then the following conditions are equivalent:*

(i) The set $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(ii) There exist $A,B>0$ such that the spectrum of almost every Ron-Shen matrix $G(x)$ is contained in the interval $[A,B]$:

$$\sigma(G(x))\subseteq[A,B] \qquad a.a.\ x\in\mathbb{R}^d\,.$$

(iii) There exist $A,B>0$ such that

$$A\|c\|_2^2 \le \sum_{j\in\mathbb{Z}^d}\Big|\sum_{k\in\mathbb{Z}^d} c_k g(x+j\alpha-\tfrac{k}{\beta})\Big|^2 \le B\|c\|_2^2 \quad a.a.\ x\in\mathbb{R}^d, c\in\ell^2(\mathbb{Z}^d)\,. \tag{4}$$

(iv) There exists a so-called dual window γ, such that $\mathcal{G}(\gamma,\alpha,\beta)$ is a Bessel sequence and γ satisfies the biorthogonality condition

$$\langle\gamma, M_{l/\alpha}T_{k/\beta}g\rangle = (\alpha\beta)^d\delta_{k,0}\delta_{l,0}, \qquad \forall k,l\in\mathbb{Z}^d\,. \tag{5}$$

REMARKS: 1. Condition (iii) is only a simple reformulation of (ii), because (4) in different notation is just

$$\|P(x)c\|_2^2 = \langle P(x)c, P(x)c\rangle = \langle G(x)c, c\rangle \asymp \|c\|_2^2 .$$

2. Condition (iii) as stated has been used successfully by Janssen [19, 21]. The construction of a dual window is underlying the proofs that $\mathcal{G}(g,\alpha,\beta)$ is a frame for the Gaussian $g(t) = e^{-\pi t^2}$ in [16] and for the one-sided exponential $g(t) = e^{-t}\chi_{\mathbb{R}^+}(t)$ [19].

3. Condition (iii) establishes a fundamental link between Gabor analysis and the theory of sampling in shift-invariant spaces. It says that the set $x+\alpha\mathbb{Z}$ is a set of sampling for the shift-invariant space $V_h(g)$ with generator g and step-size $h = 1/\beta$. Thus $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$, if and only if each set $x+\alpha\mathbb{Z}$ is a set of sampling for $V(g)$ *with uniform constants independent of* $x \in \mathbb{R}^d$.

In our analysis we will use another reformulation of condition (iii).

Lemma 4. *Let $g \in L^2(\mathbb{R}^d)$ and $\alpha, \beta > 0$. Then the following conditions are equivalent:*

(i) The set $\mathcal{G}(g,\alpha,\beta)$ is a frame for $L^2(\mathbb{R}^d)$.

(v) The set of the pre-Gramians $\{P(x)\}$ is uniformly bounded on $\ell^2(\mathbb{Z}^d)$, and possess a uniformly bounded set of left-inverses, i.e., there exist matrices $\Gamma(x), x \in \mathbb{R}^d$, such that

$$\Gamma(x)P(x) = I \qquad a.a.\ x \in \mathbb{R}^d , \tag{6}$$

$$\|\Gamma(x)\|_{op} \le C \qquad a.a.\ x \in \mathbb{R}^d . \tag{7}$$

In this case, the function γ defined by $\gamma(x+\alpha j) = \beta^d \bar{\Gamma}_{0,j}(x)$, where $x \in [0,\alpha)^d$ and $j \in \mathbb{Z}^d$, or equivalently

$$\gamma(x) = \beta^d \sum_{j\in\mathbb{Z}^d} \bar{\Gamma}_{0,j}(x)\chi_{[0,\alpha)^d}(x-\alpha j) , \quad x \in \mathbb{R}^d, \tag{8}$$

satisfies the biorthogonality condition (5).

Proof. (**i**) $\Rightarrow$ (**v**). If $\mathcal{G}(g,\alpha,\beta)$ is a frame, then by Theorem 3(ii)

$$\sum_{j\in\mathbb{Z}^d} \Big| \sum_{k\in\mathbb{Z}^d} c_k g(x+j\alpha - \tfrac{k}{\beta})\Big|^2 = \langle P(x)c, P(x)c\rangle = \langle G(x)c, c\rangle \asymp \|c\|_2^2 ,$$

with bounds independent of x. Consequently $G(x)$ is bounded and invertible on $\ell^2(\mathbb{Z}^d)$. Therefore the operators $P(x)$ are uniformly bounded on $\ell^2(\mathbb{Z}^d)$, and we can define $\Gamma(x) = G(x)^{-1}P^*(x)$. Then

$$\Gamma(x)P(x) = \big((P^*(x)P(x))^{-1}P^*(x)\big)P(x) = \mathrm{Id},$$

and

$$\|\Gamma(x)\|_{op} \le \|G(x)^{-1}\|_{op}\|P(x)\|_{op} \le A^{-1}B^{1/2}.$$

(**v**) $\Rightarrow$ (**ii**). Conversely, if $P(x)$ possesses a bounded left-inverse $\Gamma(x)$, then

$$\|c\|_2^2 = \|\Gamma(x)P(x)c\|_2^2 \le \|\Gamma(x)\|_{op}^2\|P(x)c\|_2^2 \le C^2\langle G(x)c, c\rangle \le C^2\|P(x)\|_{op}^2\|c\|_2^2 ,$$

and this implies condition (**ii**) of Theorem 3.

We next verify that γ as defined in (8) satisfies the biorthogonality condition (5):

$$
\begin{aligned}
\langle \gamma, M_{l/\alpha}T_{k/\beta}g\rangle &= \int_{\mathbb{R}^d} \gamma(x)\bar{g}(x-k/\beta)\, e^{-2\pi i l\cdot x/\alpha}\, dx \\
&= \int_{[0,\alpha]^d} \sum_{j\in\mathbb{Z}^d} \gamma(x+\alpha j)\bar{g}(x+\alpha j-k/\beta)\ e^{-2\pi i l\cdot x/\alpha}\, dx \\
&= \beta^d \int_{[0,\alpha]^d} \sum_{j\in\mathbb{Z}^d} \bar{\Gamma}_{0,j}(x)\bar{g}(x+\alpha j-k/\beta)\ e^{-2\pi i l\cdot x/\alpha}\, dx \qquad (9)\\
&= \beta^d \int_{[0,\alpha]^d} \delta_{k,0} e^{-2\pi i l\cdot x/\alpha}\, dx = (\alpha\beta)^d \delta_{k,0}\delta_{l,0}\,.
\end{aligned}
$$

■

The function γ in (8) is a dual window of g, as defined in condition (iv) of Theorem 3, provided that $\mathcal{G}(\gamma,\alpha,\beta)$ is a Bessel sequence. The following result gives a sufficient condition.

Lemma 5. *Assume that there exists a (Lebesgue measurable) vector-valued function $\sigma(x)$ from $\mathbb{R}^d$ to $\ell^2(\mathbb{Z}^d)$ with period α, such that*

$$
\sum_{j\in\mathbb{Z}^d} \sigma_j(x)\bar{g}(x+\alpha j-\tfrac{k}{\beta}) = \delta_{k,0} \qquad a.a.\ x\in\mathbb{R}^d\,. \tag{10}
$$

If $\sum_{j\in\mathbb{Z}^d}\sup_{x\in[0,\alpha]^d}|\sigma_j(x)|<\infty$, then $\mathcal{G}(g,\alpha,\beta)$ is a frame. Moreover, with

$$
\gamma(x) = \beta^d \sum_{j\in\mathbb{Z}^d} \sigma_j(x)\chi_{[0,\alpha)^d}(x-\alpha j), \qquad x\in\mathbb{R}^d,
$$

the set $\mathcal{G}(\gamma,\alpha,\beta)$ is a dual frame of $\mathcal{G}(g,\alpha,\beta)$.

Proof. The computation in (9) shows that γ satisfies the biorthogonality condition (5). The additional assumption implies that

$$
\sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,\alpha]^d} |\gamma(x+\alpha k)| < \infty\,.
$$

Consequently, γ is in the amalgam space $W(\ell^1)$. This property guarantees that $\mathcal{G}(\gamma,\alpha,\beta)$ is a Bessel system [33]. Thus condition (*iv*) of Theorem 3 is satisfied, and $\mathcal{G}(g,\alpha,\beta)$ is a frame. The biorthogonality condition (5) implies that $\mathcal{G}(\gamma,\alpha,\beta)$ is a dual frame of $\mathcal{G}(g,\alpha,\beta)$.

■

2.2. **Totally Positive Functions.** The notion of totally positive functions was introduced in 1947 by I. J. Schoenberg [27]. A non-constant measurable function $g:\mathbb{R}\to\mathbb{R}$ is said to be *totally positive*, if it satisfies the following condition: For every two sets of increasing real numbers

$$
x_1<x_2<\cdots<x_N, \qquad y_1<y_2<\cdots<y_N, \qquad N\in\mathbb{N}, \tag{11}
$$

we have the inequality

$$D = \det\left[g(x_j - y_k)\right]_{1\le j,k\le n} \geq 0. \tag{12}$$

Schoenberg [28] connected the total positivity to a factorization of the (two-sided) Laplace transform of g,

$$\mathcal{L}[g](s) = \int_{-\infty}^{\infty} e^{-st} g(t)\, dt =: \frac{1}{\Phi(s)}.$$

Theorem 6 (Schoenberg [27])**.** *The function $g : \mathbb{R} \to \mathbb{R}$ is totally positive, if and only if its two-sided Laplace transform exists in a strip $S = \{s \in \mathbb{C} : \alpha < Re\, s < \beta\}$, and*

$$\Phi(s) = \frac{1}{\mathcal{L}[g](s)} = Ce^{-\gamma s^2 + \delta s} \prod_{\nu=1}^{\infty} (1 + \delta_\nu s) e^{-\delta_\nu s}, \tag{13}$$

with real parameters $C, \gamma, \delta, \delta_\nu$ satisfying

$$C > 0, \quad \gamma \geq 0, \quad 0 < \gamma + \sum_{\nu=1}^{\infty} \delta_\nu^2 < \infty. \tag{14}$$

A comprehensive study of total positivity is given in the book of Karlin [23]. It is known that, if g is totally positive and integrable, then g decays exponentially (see [28, p. 340]). In this article, we restrict our attention to the class of totally positive functions $g \in L^1(\mathbb{R})$ with the factorization

$$\Phi(s) = \frac{1}{\mathcal{L}[g](s)} = Ce^{\delta s} \prod_{\nu=1}^{M} (1 + \delta_\nu s)\,, \tag{15}$$

for $M \in \mathbb{N}$ and real δ_ν. This means that the denominator of $\mathcal{L}[g]$ has only finitely many roots. Equivalently, the Fourier transform of g can be extended to a meromorphic function with a finite number of poles on the imaginary axis and no other poles. As noted in [29, p. 247], the exponential factor can be omitted, as it corresponds to a simple shift of g. In the following we will call a totally positive function satisfying (15) *totally positive of finite type* and refer to M as the type of g.

Schoenberg and Whitney [29] gave a complete characterization of the case when the determinant D in (12) satisfies $D > 0$.

Theorem 7 ([29])**.** *Let $g \in L^1(\mathbb{R})$ be a totally-positive function of finite type. Furthermore, let m be the number of positive δ_ν and n be the number of negative δ_ν in* (15)*, and $m + n \geq 2$. For a set of points in* (11)*, the determinant $D = \det[g(x_j - y_k)]_{j,k=1,\dots,N}$ is strictly positive, if and only if*

$$x_{j-m} < y_j < x_{j+n} \qquad \textit{for } 1 \leq j \leq N. \tag{16}$$

Here, we use the convention that $x_j = -\infty$, if $j < 1$, and $x_j = \infty$, if $j > N$.

The conditions in (16) are nowadays called the Schoenberg-Whitney conditions for g. They have been used extensively in the analysis of spline interpolation by Schoenberg and others (see the monograph [30]). They will be crucial for our construction of a left inverse of the pre-Gramian matrix in (2).

As a generalization of the pre-Gramians in (2), we consider bi-infinite matrices of the form

$$P = [g(x_j - y_k)]_{j,k\in\mathbb{Z}}\,, \tag{17}$$

where each sequence $X = (x_j)_{j\in\mathbb{Z}}$ and $Y = (y_k)_{k\in\mathbb{Z}} \subseteq \mathbb{R}$ is strictly increasing. Moreover, the sequence $(x_j)_{j\in\mathbb{Z}}$ is supposed to be denser in the sense of the following condition:

$$(C_r)\quad \begin{cases} \text{(a) every interval } (y_k, y_{k+1}) \text{ contains at least one point } x_j; \\ \text{(b) there is an } r\in\mathbb{N} \text{ such that } |(y_k, y_{k+r})\cap X| \geq r+1 \text{ for all } k. \end{cases}$$

Our main tool for the study of Gabor frames and sampling theorems will be the following technical result. It can be interpreted as a suitable extension of total positivity to infinite matrices.

Theorem 8. *Let $g \in L^1(\mathbb{R})$ be a totally-positive function of finite type. Let m be the number of positive δ_ν, n be the number of negative δ_ν in (15), and $M = m+n \geq 1$. Assume that the sequences $(x_j)_{j\in\mathbb{Z}}$ and $(y_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ satisfy condition (C_r).*

Then the matrix $P = [g(x_j - y_k)]_{j,k\in\mathbb{Z}}$ defines a bounded operator on $\ell_2(\mathbb{Z})$. It has an algebraic left-inverse $\Gamma = [\gamma_{k,j}]_{k,j\in\mathbb{Z}}$, and

$$\gamma_{k,j} = 0, \qquad \textit{if} \quad x_j < y_{k-rm} \;\textit{ or }\; x_j > y_{k+rn}. \tag{18}$$

Proof. We construct a left-inverse Γ with the desired properties by defining each row of Γ separately. It suffices to consider the row with index $k = 0$, as the construction of all other rows is done in the same way. The goal of the first three steps is to select a finite subset of x_j's and y_k's that satisfy the Schoenberg-Whitney conditions. (Our choice of indices is not unique and not symmetric in m and n, it minimizes the number of case distinctions.)

Step 1: Column selection. First, consider the case $m, n > 0$ and set $N := (m+n-1)(r+1)$, if $n > 1$, and $N = m(r+1)+1$, if $n = 1$. We define an $N \times N$ submatrix P_0 of P in the following way. As columns of P_0, we select columns of P between

$$k_1 = -(r+1)m + 1 \quad \text{ and } \quad k_2 = k_1 + N - 1\,.$$

Hence $k_2 = (r+1)(n-1)$ for $n > 1$, and $k_2 = 1$ for $n = 1$. For later purposes, note that $k_1 \leq -m < 0 < n \leq k_2$. Therefore, the column with index $k = 0$ has at least m columns to its left and n columns to its right.

Step 2: Selection of a square matrix. Assumption (C_r) and our definition of N imply that the interval $I = (y_{k_1+m-1}, y_{k_2-n+1})$ contains at least N points x_j. More precisely, for $n > 1$ we write

$$(y_{k_1+m-1}, y_{k_2-n+1}) = (y_{-rm}, y_{r(n-1)}) = \bigcup_{\nu=-m}^{n-2} (y_{r\nu}, y_{r(\nu+1)})$$

and find at least $r+1$ points x_j in each subinterval $(y_{r\nu}, y_{r(\nu+1)})$ with $-m \leq \nu \leq n-2$. This amounts to at least $(m+n-1)(r+1) = N$ points in I. If $n = 1$, we have $(y_{k_1+m-1}, y_{k_2-n+1}) = (y_{-rm}, y_1)$ and find $m(r+1)$ points x_j in (y_{-rm}, y_0) plus at least one additional point in (y_0, y_1).

We let

$$j_1 := \min\{j : x_j > y_{k_1+m-1}\}, \qquad j_2 := \max\{j : x_j < y_{k_2-n+1}\}\,. \tag{19}$$

We have just shown that the set

$$X_0 = \{x_j : j_1 \le j \le j_2\} \subset (y_{k_1+m-1}, y_{k_2-n+1})$$

contains at least N elements. We choose a subset

$$X_0' = \{\xi_1 < \cdots < \xi_N\} \subset X_0,$$

that contains precisely N elements and satisfies

$$(y_k, y_{k+1}) \cap X_0' \neq \emptyset \quad \text{for} \quad k_1 + m - 1 \le k \le k_2 - n.$$

That is, we choose one point x_j in each interval (y_k, y_{k+1}), with $k_1 + m - 1 \le k \le k_2 - n$, and an additional $n + m - 1$ points $x_j \in (y_{k_1+m-1}, y_{k_2-n+1})$. Note that

$$y_{k_1+m-1} < \xi_1 = \min X_0' < y_{k_1+m} < y_{k_2-n} < \max X_0' = \xi_N < y_{k_2-n+1}. \tag{20}$$

Now set

$$\eta_k = y_{k_1+k-1}, \qquad 1 \le k \le N\,,$$

and define the matrix

$$P_0 = (g(\xi_j - \eta_k))_{j,k=1,\dots,N}.$$

Then P_0 is a quadratic $N \times N$-submatrix of P.

Step 3: Verification of the Schoenberg-Whitney conditions. We next show that P_0 is invertible by checking the Schoenberg-Whitney conditions. First, by (20), we have

$$\xi_1 = \min X_0' < y_{k_1+m} = \eta_{m+1}.$$

By the construction of X_0', this inequality progresses from left to right, i.e.,

$$\xi_j < y_{k_1+m-1+j} = \eta_{j+m} \qquad \text{for} \quad 1 \le j \le N - m.$$

Likewise, we also have

$$\eta_{N-n} = y_{k_2-n} < \max X_0' = \xi_N,$$

and this inequality progresses from right to left, i.e.,

$$\eta_j < \xi_{j+n} \qquad \text{for} \quad 1 \le j \le N - n.$$

Therefore, the Schoenberg-Whitney conditions (16) are satisfied, and Theorem 7 implies that $\det P_0 > 0$.

Step 4: Linear dependence of the remaining columns of P. We now make some important observations.

Choose indices k_0 and $s \in \mathbb{Z}$ with $k_0 < k_1$ and $m < s \le N$, and consider the new set $\{\eta_k' : k = 1, \dots, N\}$ consisting of the points

$$y_{k_0} < y_{k_1} < \cdots < y_{k_1+s-2} < y_{k_1+s} < \cdots < y_{k_2}$$

and the corresponding $N \times N$-matrix $P_0' = (g(\xi_j - \eta_k'))_{j,k=1,\dots,N}$. This matrix is obtained from P_0 by adding the column $\big(g(\xi_j - y_{k_0})\big)_{1\le j\le N}$ as its first column and

deleting the column $\big(g(\xi_j - \eta_s)\big)_{1\le j\le N}$. Then η_m appears in the $m+1$-st column of P_0'. By (20), we see that

$$\eta'_{m+1} = \eta_m = y_{k_1+m-1} < \xi_1 .$$

Consequently, the Schoenberg-Whitney conditions are violated and therefore $\det P_0' = 0$ by Theorem 7. Since this holds for all $s > m$, the vector $(g(\xi_j - y_{k_0}))_{1\le j\le N}$ must be in the linear span of the first m columns of P_0, namely $(g(\xi_j - y_k))_{1\le j\le N}$ for $k = k_1, , \dots k_1 + m - 1$.

Likewise, choose $k_3 > k_2$, $1 \le s \le N - n$, and consider the new set $\{\eta''_k : k = 1, \dots, N\}$ consisting of the points

$$y_{k_1} < \cdots < y_{k_1+s-2} < y_{k_1+s} < \cdots < y_{k_2} < y_{k_3}$$

and the corresponding $N \times N$-matrix $P''_0 = (g(\xi_j - \eta''_k))_{j,k=1,\dots,N}$. This matrix is obtained from P_0 by adding the column $(g(\xi_j - y_{k_3}))_{1\le j\le N}$ as its last (=N-th) column and deleting the column $\big(g(\xi_j - \eta_s)\big)_{1\le j\le N}$. Then η_{N-n+1} appears in the $n+1$-st column of P''_0, counted from right to left, and

$$\eta''_{N-n} = \eta_{N-n+1} = y_{k_1+N-n} = y_{k_2-n+1} > \xi_N$$

by (20). Again the Schoenberg-Whitney conditions are violated and therefore $\det P''_0 = 0$. We conclude that the vector $(g(\xi_j - y_{k_3}))_{1\le j\le N}$ must lie in the linear span of the last n columns of P_0.

Step 5: Construction of the left-inverse. Recall that $k_1 = -m(r+1)+1$ and that $\eta_{(r+1)m} = y_{k_1+(r+1)m-1} = y_0$. Let c^T denote the $(r+1)m$-th row vector of P_0^{-1}. By definition of the inverse, we have $\sum_{j=1}^N c_j g(\xi_j - \eta_k) = \delta_{k,(r+1)m}$, or equivalently, for $k_1 \le k \le k_2$,

$$\sum_{j=1}^{N} c_j g(\xi_j - y_k) = \delta_{k,0}\,. \tag{21}$$

Let us now consider the other columns with $k < k_1$ or $k > k_2$. Since $k_1 \le -m < 0 < n \le k_2$ and every vector $(g(\xi_j - y_k))_{1\le j\le N}$ lies either in the span of the first m columns of P_0 (for $k < k_1$) or in the span of the last n columns of P_0 (for $k > k_2$), we obtain that

$$\sum_{j=1}^{N} c_j g(\xi_j - y_k) = 0\,.$$

Therefore, the identity (21) holds for all $k \in \mathbb{Z}$.

Next we fill the vector c with zeros and define the infinite vector γ_0 by

$$\gamma_{0,j} = \begin{cases} c_{j'} & \text{if } x_j = \xi_{j'} \in X'_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\sum_{j\in\mathbb{Z}} \gamma_{0,j}\, g(x_j - y_k) = \sum_{j=1}^{N} c_j g(\xi_j - y_k) = \delta_{k,0}\,.$$

Thus γ_0 is a row of the left-inverse of P. By construction, γ_0 has at most N non-zero entries. In particular, if $x_j < y_{k_1+m-1} = y_{-rm}$, then we have $j < j_1$ and thus $\gamma_{0,j} = 0$. Similarly, if $x_j > y_{k_2-n+1} = y_{r(n-1)}$ for $n > 1$, and $x_j > y_{k_2-n+1} = y_1$ for $n = 1$, then we have $j > j_2$ and $\gamma_{0,j} = 0$. This gives the support properties of the entries $\gamma_{0,k}$ of row $k = 0$ of the left-inverse Γ.

Step 6: The other rows of Γ. The construction of the k-th row of Γ is similar. We choose the columns between $k_1 = k-(r+1)m+1$ and $k_2 = k_1 + N - 1$ of P, and, accordingly, we choose suitable rows between the indices $j_1 = \min\{j : x_j > y_{k_1+m-1}\}$ and $j_2 := \max\{j : x_j < y_{k_2-n+1}\}$. Then the column of P containing y_k has at least m columns to its left and n columns to its right. We then proceed to select ξ_j's and define an $N \times N$-matrix P_k and verify that $\det P_k > 0$. The k-th row of Γ is obtained by padding the appropriate row (with row-index $(r+1)m$) of P_k^{-1} with zeros. By this construction one obtains a vector $\gamma_k = (\gamma_{k,j})_{j\in\mathbb{Z}}$, for which

$$\sum_{j\in\mathbb{Z}} \gamma_{k,j} g(x_j - y_l) = \delta_{k,l}$$

holds. Furthermore, $\gamma_{k,j} = 0$ when $x_j < y_{k-rm}$ and when $x_j > y_{k+rn}$.

Step 7: The remaining cases. The cases where $m = 0$ or $n = 0$ are simple adaptations of the above steps. For $m = 0$, $n \geq 2$ we choose $N = (n-1)(r+1)$. The indices for the submatrix P_k that occurs in the construction of the k-th row of Γ are $k_1 = k$, $k_2 = k + N - 1$, $j_1 = j_1(k) = j_1 = \max\{j : x_j < y_{k_1}\}$ and $j_2 = j_2(k) = \max\{j : x_j < y_{k_2}\}$. Now we proceed as before.

We note that Step 4 simplifies a bit. For $m = 0$ the function g is supported on $(-\infty, 0)$ by [28, p. 339]. Consequently, if $k_0 < k_1$ and $j \geq j_1$, then $x_j - y_{k_0} > 0$ and $g(x_j - y_{k_0}) = 0$.

Thus the column vectors of P to the left of the submatrix P_0 are identically zero and no further proof is needed for linear dependence.

The case $n = 0$ is similar. It can be reduced to the previous case by a reflection $x \to -x$, which interchanges the role of m and n.

The special case of $m = 0$ and $n = 1$ can be solved by taking $N = 2$, $k_1 = k$, $k_2 = k+1$, $j_1 = \max\{j : x_j < y_k\}$, $j_2 = \min\{j : x_j > y_k\}$, and the 2×2-matrix

$$P_k = \begin{pmatrix} g(x_{j_1} - y_k) & g(x_{j_1} - y_{k+1}) \\ g(x_{j_2} - y_k) & g(x_{j_2} - y_{k+1}) \end{pmatrix}.$$

In this simple case the size of the matrix is independent of the parameter r occurring in condition C_r. ∎

3. Gabor Frames with Totally Positive functions

In this section we prove the main result about Gabor frames. Recall that a totally positive function is said to be of finite type, if its two-sided Laplace transform factors as $\mathcal{L}[g](s)^{-1} = Ce^{\delta s}\prod_{\nu=1}^{M}(1+\delta_\nu s)$ with real numbers δ, δ_ν.

Theorem 9. *Assume that g is a totally positive function of finite type and $M = m + n \geq 2$, where m is the number of positive zeros and n the number of negative*

zeros of $1/\mathcal{L}(g)$. *Then* $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$. *The Gabor frame* $\mathcal{G}(g,\alpha,\beta)$ *possesses a piecewise continuous dual window* γ *with compact support in* $[-\frac{rm}{\beta} - \alpha, \frac{rn}{\beta} + \alpha]$, *where* $r := \lfloor \frac{1}{1-\alpha\beta} \rfloor$.

By taking a Fourier transform, we obtain the following corollary.

Corollary 10. *If* $h(\tau) = C\prod_{\nu=1}^{M}(1+2\pi i\delta_\nu\tau)^{-1}$ *for* $M \geq 2$, *then* $\mathcal{F}(h) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$ *and* $\mathcal{G}(h,\alpha,\beta)$ *possesses a bandlimited dual window* θ *with* $\operatorname{supp}\hat{\theta} \subseteq [-\frac{rm}{\alpha} - \beta, \frac{rn}{\alpha} + \beta]$.

REMARK: We have excluded the case $M = 1$ for the formulation of the theorem, because it corresponds to the one-sided exponential $g(t) = e^{-t}\chi_{\mathbb{R}_+}(t)$. This function is discontinuous and therefore is not subject to the Balian-Low principle. Instead, we have only $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta \leq 1\}$.

Proof of Theorem 9. Since $g \in L^1(\mathbb{R})$ is totally positive, it decays exponentially, and since $M \geq 2$, its Fourier transform $\hat{g}(\xi) = Ce^{2\pi i\delta\xi}\prod_{\nu=1}^{M}(1+2\pi i\delta_\nu\xi)^{-1}$ decays at least like $|\hat{g}(\xi)| \leq \tilde{C}(1+\xi^2)^{-1}$. In particular, g is continuous. As a consequence, the assumptions of the Balian-Low theorem are satisfied [4, 14] and $\mathcal{F}(g) \subseteq \{(\alpha,\beta) \in \mathbb{R}_+^2 : \alpha\beta < 1\}$.

To prove that $\mathcal{G}(g,\alpha,\beta)$ is a frame for $\alpha\beta < 1$, we will construct a family of uniformly bounded left-inverses for the pre-Gramians $P(x)$ of g and then use Lemma 4.

Fix $x \in [0,\alpha]$ and consider the sequences $x_j = x+\alpha j$ and $y_k = k/\beta$, $j,k \in \mathbb{Z}$. We first check condition (C_r). By our assumption, we have $\alpha < 1/\beta$ and every interval $(k/\beta, (k+1)/\beta)$ contains at least one point $x+\alpha j$. Every interval $(k/\beta, (k+r)/\beta)$, with $r \in \mathbb{N}$, contains at least $r+1$ points $x+\alpha j$, if $r/\beta > (r+1)\alpha$, i.e., we have

$$r > \frac{\alpha\beta}{1-\alpha\beta}, \quad \text{or equivalently} \quad r \geq \left\lfloor \frac{1}{1-\alpha\beta} \right\rfloor.$$

Consequently, condition (C_r) is satisfied with $r = \lfloor \frac{1}{1-\alpha\beta} \rfloor$. By Theorem 8, each pre-Gramian $P(x)$ with entries $g(x+\alpha j - k/\beta)$ possesses a left-inverse $\Gamma(x)$.

To apply Lemma 5, we need to show that $\Gamma(x), x \in [0,\alpha]$, is a uniformly bounded set of operators on $\ell^2(\mathbb{Z})$.

Let $P_0(x)$ be the $N \times N$-square submatrix constructed in Steps 1 and 2. The column indices k_1 and k_2 depend only on the type of g, but not on x. The row indices $j_1 = j_1(x) = \min\{j : x_j > y_{k_1+m-1}\} = \min\{j : x+\alpha j > (k_1+m-1)/\beta\}$ and $j_2 = j_2(x)$ are locally constant in x. Likewise the indices that determine which rows $j, j_1 < j < j_2$ of $P(x)$ are contained in $P_0(x)$ are locally constant. Consequently, for every $x \in [0,\alpha]$ there is a neighborhood U_x such that indices used for $P_0(y)_{jk} = g(y+\xi_j - k/\beta)$ do not depend on $y \in U_x$. Since g is continuous, $P_0(y)$ is continuous on U_x, and since $\det P_0(x) > 0$ there exists a neighborhood $V_x \subseteq U_x$, such that $\det P_0(y) \geq \det P_0(x)/2$ for $y \in V_x$.

We now cover $[0,\alpha]$ with finitely many neighborhoods V_{x_q} and obtain that $\det P_0(y) \geq \min_q \det P_0(x_q)/2 = \delta > 0$ for all $y \in [0,\alpha]$. Since each entry of the inverse matrix $P_0(y)^{-1}$ can be calculated by Cramer's rule, these entries must

be bounded by $C \det P_0(y)^{-1} \leq C\delta^{-1}$ with a constant C depending only on $\|g\|_\infty$ and the dimension N of $P_0(y)$.

By construction (Step 5), the zero-th row $\gamma(x) = (\gamma_{0,j}(x))_{j\in\mathbb{Z}}$ of the left-inverse $\Gamma(x)$ contains at most $N \leq (r+1)M$ non-zero entries, namely those of the row of $P_0(x)^{-1}$ corresponding to $y_0 = 0$. We have thus constructed a vector-valued function $x \to \gamma(x)$ from $[0,\alpha] \to \ell^\infty(\mathbb{Z})$ with the following properties:

(i) $\gamma(x)$ is piecewise continuous,
(ii) $\operatorname{card}(\operatorname{supp} \gamma(x)) \leq (r+1)M$, where according to (18)

$$\operatorname{supp} \gamma(x) = \{j : \gamma_{0,j}(x) \not\equiv 0\} \subseteq \{j : \frac{-rm}{\beta} \leq x+\alpha j \leq \frac{rn}{\beta}\} \subseteq \{j : \frac{-rm}{\alpha\beta} - 1 \leq j \leq \frac{rn}{\alpha\beta}\},$$

(iii) and $\sup_{x\in[0,\alpha]} \|\gamma(x)\|_\infty = C < \infty$.

Consequently, the dual window $\gamma(x) = \beta \sum_{j\in\mathbb{Z}^d} \overline{\gamma}_{0,j}(x)\chi_{[0,\alpha)}(x-\alpha j)$ corresponding to $\Gamma(x)$ by Lemma 5, has compact support on the interval $[-\frac{rm}{\beta} - \alpha, \frac{rn}{\beta} + \alpha]$, is piecewise continuous, and is bounded. In particular, it satisfies the Bessel property (see [33] or [14, Cor. 6.2.3]).

We have constructed a dual window for $\mathcal{G}(g,\alpha,\beta)$ satisfying the Bessel property. By Theorem 3 and Lemma 4, $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame. ■

3.1. **Remarks and Conjectures.** In the proof of Theorem 9 we have constructed a compactly supported dual window γ for $\mathcal{G}(g,\alpha,\beta)$. This construction is explicit and can be realized numerically, because it requires only the inversion of finite matrices. To determine the values $\gamma(x+j\alpha)$, one has to solve the linear $N \times N$ system $P_0(x)\gamma(x) = e$ for a vector e of the standard basis of $\mathbb{R}^N$.

We observe that the canonical dual window (provided by standard frame theory) has better smoothness properties. The regularity theory for the Gabor frame operator implies that the canonical dual window γ° decays exponentially and its Fourier transform $\widehat{\gamma^\circ}(\xi)$ decays like $\mathcal{O}(|\xi|^{-M})$, where M is the type of g. See [12,14,15,32].

Theorem 9 raises many new questions. Theorem 9 suggests the natural conjecture that the frame set of *every* totally positive continuous function g in $L^1(\mathbb{R})$ is $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$. Our proof is tailored to totally positive functions of finite type, most likely the proof of the conjecture will require a different method.

In a larger context one may speculate about the set $\mathcal{M}$ of functions such that the frame set is exactly $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$. In other words, when is the necessary density condition $\alpha\beta < 1$ also sufficient for $\mathcal{G}(g,\alpha,\beta)$ to be a frame? The invariance properties of Gabor frames imply that the class $\mathcal{M}$ must be invariant under time-frequency shifts, dilations, involution, and the Fourier transform. Furthermore, if $\mathcal{F}(g) = \{(\alpha,\beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$, then both g and $\hat{g}$ must have infinite support.

A general method for constructing functions in $\mathcal{M}$ can be extracted from [22]. We write $\hat{c}(\xi) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi i k\xi}$ for the Fourier series of a sequence (c_k) and then

define, for a given function $g_0 \in L^2(\mathbb{R})$,

$$C_{g_0} = \{f \in L^2(\mathbb{R}) : f = \sum_{k,l\in\mathbb{Z}} c_k d_l T_k M_l g_0, c, d \in \ell^1(\mathbb{Z}), \inf_\xi(|\hat{c}(\xi)\, \hat{d}(\xi)| > 0\} .$$

Lemma 11. *Let g_0 be a totally positive function of finite type $M \geq 2$. If $g \in C_{g_0}$, then the frame set of g is $\mathcal{F}(g) = \{(\alpha, \beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$.*

This trick was used by Janssen and Strohmer in [22]. They showed that the hyperbolic secant $g_1(t) = (e^t + e^{-t})^{-1}$ belongs to the set C_φ for the Gaussian window $\varphi(t) = e^{-\pi t^2}$ and then concluded that $\mathcal{F}(g_1) = \{(\alpha, \beta) \in \mathbb{R}^2_+ : \alpha\beta < 1\}$. The general argument is identical.

Each class C_g is completely determined by the zeros of the Zak transform of g. Let $Zg(x, \xi) = \sum_{k\in\mathbb{Z}} g(t-k)e^{2\pi ik\xi}$ be the Zak transform of g. Since $Z(T_k M_l g)(x, \xi) = e^{2\pi i(lx-k\xi)} Zg(x, \xi)$, every $g \in C_{g_0}$ has a Zak transform of the form

$$Zg(x, \xi) = \hat{c}(-\xi)\hat{d}(x) Zg_0(x, \xi) .$$

The definition of C_{g_0} implies that Zg and Zg_0 have the same zero set. If g_0 and h are two totally positive functions, then the zero sets of Zg_0 and Zh are different in general, therefore Lemma 11 leads to distinct sets C_{g_0}. The zeros of the Zak transform seems to be some kind of invariant for the Gabor frame problem, but their deeper significance is still mysterious.

4. Sampling Theorems

In this section we exploit the connection between Gabor frames and sampling theorems and prove new and sharp sampling theorems for shift-invariant spaces. Originally shift-invariant spaces were used as a substitute for bandlimited functions and were defined as the span of integer translates of a given function g. We refer to the survey [3] for the theory of sampling in shift-invariant spaces. We will deal with a slightly more general class of spaces that are generated by arbitrary shifts.

Let $Y = (y_k)_{k\in\mathbb{Z}}$ be a strictly increasing sequence and consider the quasi shift-invariant space

$$V_Y(g) = \{f \in L^2(\mathbb{R}) : f = \sum_{k\in\mathbb{Z}} c_k g(. - y_k)\} .$$

We require that the sequence $Y = (y_k)$ of shift parameters satisfies the conditions

$$0 < q_Y = \inf_k (y_{k+1} - y_k) \leq \sup_k (y_{k+1} - y_k) = Q_Y < \infty. \tag{22}$$

Such sequences are called quasi-uniform or uniformly discrete. The numbers $Q_Y, q_Y > 0$ are the mesh-norm and the separation distance of Y.

For the norm equivalence $\|f\|_2 \asymp \|c\|_2$ for $f \in V_Y(g)$ we need that $\{g(. - y_k : y_k \in Y\}$ is a Riesz basis for $V_Y(g)$.

Lemma 12. *Let g be an arbitrary totally positive function.*

(i) If $Y = h\mathbb{Z}$ with $h > 0$, then $\{g(. - hk), k \in \mathbb{Z}\}$ is a Riesz basis for $V_Y(g)$.

(ii) If Y is quasi-uniform, then $\{g(. - y_k) : k \in \mathbb{Z}\}$ is a Riesz basis for $V_Y(g)$.

Proof. (i) Since $\hat{g}$ is continuous, does not have any real zeros, and $\hat{g}(\xi)$ decays at least like $C/|\xi|$, every periodization of $|\hat{g}|^2$ is bounded above and below. This property is equivalent to the Riesz basis property, e.g. [7, Thm. 7.2.3].

Of course, (i) also follows from (ii).

(ii) For the general case we use Zygmund's inequality [35, Thm. 9.1]: If I is an interval of length $|I| > \frac{1+\delta}{q_Y}$, then

$$\int_I \Big| \sum_k c_k e^{2\pi i y_k \xi} \Big|^2 \, d\xi \geq A_\delta |I| \|c\|_2^2$$

for a constant depending only on $\delta > 0$.

If $f = \sum_k c_k g(. - y_k)$, then

$$\begin{aligned} \|f\|_2^2 = \|\hat{f}\|_2^2 &= \int_{\mathbb{R}} \Big| \sum_k c_k e^{-2\pi i y_k \tau} \Big|^2 |\hat{g}(\tau)|^2 \, d\tau \\ &\geq \inf_{\tau \in I} |\hat{g}(\tau)|^2 \int_I \Big| \sum_k c_k e^{-2\pi i y_k \tau} \Big|^2 \, d\xi \\ &\geq C|I| A_\delta \|c\|_2^2 . \end{aligned}$$

Here $\inf_{\tau \in I} |\hat{g}(\tau)|^2 > 0$, because $\hat{g}$ does not have any real zeros by Theorem 6. ∎

We are interested to derive sampling theorems for generalized shift-invariant spaces that are generated by a totally positive function g. Our goal is to construct strictly increasing sequences $X = (x_j)$ that yield a sampling inequality

$$A\|f\|_2^2 \leq \sum_{j \in \mathbb{Z}} |f(x_j)|^2 \leq B\|f\|_2^2 \qquad \text{for all } f \in V_Y(g) \tag{23}$$

for some constants $A, B > 0$ independent of f. Following Landau [24], a set $X \subset \mathbb{R}$ that satisfies the norm equivalence (23) is called a set of (stable) sampling for $V_Y(g)$. Except for bandlimited functions and B-spline generators only qualitative results are known about sets of sampling in shift-invariant spaces.

We first give an equivalent condition for sets of sampling in $V_Y(g)$. As in Lemma 4 we obtain the following characterization of sets of sampling in $V_Y(g)$.

Lemma 13. *Let $g \in L^2(\mathbb{R})$, and let $Y = (y_k)_{k \in \mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence. Then a set $\{x_j\} \subset \mathbb{R}$ is a set of sampling for $V_Y(g)$, if and only if the pre-Gramian P with entries $p_{jk} = g(x_j - y_k)$ possesses a left-inverse Γ that is bounded on $\ell^2(\mathbb{Z})$.*

The case of uniform sampling in shift-invariant spaces is completely settled by the results in Section 3.

Corollary 14. *Let g be a totally positive function of finite type $M \geq 2$ and $Y = h\mathbb{Z}$. If $\alpha < h$ and $x \in \mathbb{R}$ is arbitrary, then the set $x + \alpha\mathbb{Z}$ is a set of sampling for $V_Y(g)$. More precisely, there exist positive constants A, B independent of x, such that*

$$A\|f\|_2^2 \leq \sum_{j \in \mathbb{Z}} |f(x + \alpha j)|^2 \leq B\|f\|_2^2 \qquad \textit{for all } f \in V_Y(g) .$$

Proof. We proved Theorem 9 by verifying the equivalent condition of Theorem 3, namely (4) stating that

$$A\|c\|_2^2 \le \sum_{j\in\mathbb{Z}^d} \Big|\sum_{k\in\mathbb{Z}^d} c_k g(x+j\alpha-hk)\Big|^2 \le B\|c\|_2^2 \quad \text{for all } \; x\in\mathbb{R}, c\in\ell^2(\mathbb{Z})\,. \tag{24}$$

Since $f\in V_Y(g)$ is of the form $f=\sum_k c_k g(.-hk)$ and $\|f\|_2 \asymp \|c\|_2$ by Lemma 12, the inequalities (24) are equivalent to the sampling inequality $\|f\|_2^2 \asymp \sum_{j\in\mathbb{Z}} |f(x+\alpha j)|^2$, and the constants are independent of x by Theorem 3. ∎

REMARK: The condition $\alpha < h$ is sharp. If $\alpha = h$, then there exists $x\in\mathbb{R}$, such that $x+\alpha\mathbb{Z}$ is not a set of sampling. This follows immediately from the Balian-Low theorem [4].

Our methods yield more general sampling theorems. On the one hand, we study non-uniform sampling sets, and on the other hand, we may treat quasi shift-invariant spaces. The auxiliary characterization of Lemma 13 gives a hint of how to proceed. If the sequences (x_j) and (y_k) satisfy condition (C_r) for some $r>0$, then by Theorem 8 the pre-Gramian matrix P possesses an algebraic left-inverse. To obtain a sampling theorem, we need to impose additional conditions on (x_j) and (y_k), so that this left-inverse is bounded on ℓ^2.

To verify the boundedness of a matrix, we will apply the following lemma which is a direct consequence of Schur's test, see, e.g., [14, Lemma 6.2.1].

Lemma 15. *Assume that $\mathbf{A}=(a_{jk})_{j,k\in\mathbb{Z}}$ is a matrix with bounded entries $|a_{jk}|\le C$ for $j,k\in\mathbb{Z}$. Furthermore, assume that there exists a strictly increasing sequence $(j_k)_{k\in\mathbb{Z}}$ of row indices $j_k\in\mathbb{Z}$ and $N\in\mathbb{N}$ such that $a_{jk}=0$ for $|j-j_k|\ge N$. Then*

$$\|\mathbf{A}\|_{\ell^2\to\ell^2} \le (2N-1)C\,. \tag{25}$$

Proof. The conditions give

$$K_2 := \sup_{k\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} |a_{jk}| = \sup_{k\in\mathbb{Z}} \sum_{j=j_k-N+1}^{j_k+N-1} |a_{jk}| \le (2N-1)C.$$

For the estimate of the column sums, we define the set

$$N_j = \{k\in\mathbb{Z} : a_{jk}\ne 0\} \subseteq \{k\in\mathbb{Z} : |j-j_k|<N\} \qquad \text{for } j\in\mathbb{Z}\,.$$

Since (j_k) is strictly increasing, N_j has at most $(2N-1)$ elements, and this gives

$$K_1 := \sup_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |a_{jk}| = \sup_{j\in\mathbb{Z}} \sum_{k\in N_j} |a_{jk}| \le (2N-1)C.$$

The assertion now follows from Schur's test. ∎

In the following we give a sufficient condition for a set X to be a set of sampling for $V_Y(g)$.

Theorem 16. *Let g be a totally positive function of finite type $M \geq 2$. Let $Y = (y_k)_{k\in\mathbb{Z}} \subset \mathbb{R}$ be an increasing quasi-uniform sequence with parameters q_Y, Q_Y defined in (22). Moreover, let $(x_j)_{j\in\mathbb{Z}} \subset \mathbb{R}$ be a strictly increasing sequence, which satisfies the following conditions:*

$$(C_r(\epsilon)) \quad \begin{cases} \text{There exist } r \in \mathbb{N},\ \epsilon \in (0, q_Y/2), \text{ and a quasi-uniform subsequence} \\ X' \subseteq X, \text{ such that} \\ \text{(a) every interval } (y_k + \epsilon, y_{k+1} - \epsilon) \text{ contains at least one point} \\ x_j \in X'; \\ \text{(b) for every } k \in \mathbb{Z}, \text{ we have } |(y_k + \epsilon, y_{k+r} - \epsilon) \cap X'| \geq r + 1. \end{cases}$$

Then X is a set of sampling for $V_Y(g)$.

Proof. **Step 1.** First, we construct a left-inverse of the pre-Gramian P as in the proof of Theorem 8 with a small modification.

We consider only the case $m > 1$, $n > 1$. For the construction of the row with the index k of the left-inverse Γ, we choose the size $N \in \mathbb{N}$ for a square submatrix P_k of P as in Step 1 and the column indices $k_1 = k - r(m+1) + 1$ and $k_2 = k_1 + N - 1$ as in Step 6.

To incorporate condition $C_r(\epsilon)$, we modify the selection of the row indices in Step 2 as follows. Assumption $C_r(\epsilon)$ and our definition of N imply that the interval $I = (y_{k_1+m-1} + \epsilon, y_{k_2-n+1} - \epsilon)$ contains at least N points $x_j \in X'$, where X' is the quasi-uniform subset of X in condition $C_r(\epsilon)$. Define

$$j_1 := \min\{j : x_j \in X',\ x_j \geq y_{k_1+m-1} + \epsilon\}, \quad j_2 := \max\{j : x_j \in X',\ x_j \leq y_{k_2-n+1} - \epsilon\},$$

then the set

$$X_k = \{x_j \in X' : j_1 \leq j \leq j_2\} \subset (y_{k_1+m-1} + \epsilon, y_{k_2-n+1} - \epsilon)$$

has at least N elements. We now choose one point $x_j \in (y_l + \epsilon, y_{l+1} - \epsilon) \cap X'$ for each $k_1 + m - 1 \leq l \leq k_2 - n$ and an additional $n + m - 1$ points $x_j \in (y_{k_1+m-1} + \epsilon, y_{k_2-n+1} - \epsilon) \cap X'$ and obtain a subset

$$X'_k = \{\xi_1 < \cdots < \xi_N\} \subseteq X_k \subset X',$$

containing precisely N elements. As before, we set

$$\eta_l = y_{k_1+l-1}, \qquad 1 \leq l \leq N,$$

and define the quadratic submatrix P_k of P by

$$P_k = (g(\xi_j - \eta_l))_{j,l=1,\dots,N}.$$

The modified construction leads to a stronger version of the Schoenberg-Whitney conditions, namely

$$\xi_j + \epsilon \leq \eta_{j+m} \text{ for } 1 \leq j \leq N - m, \qquad \eta_j + \epsilon \leq \xi_{j+n} \text{ for } 1 \leq j \leq N - n. \tag{26}$$

Step 4 remains unchanged, and the column of P with $k < k_1$ or $k > k_2$ are linearly dependent on the columns of P_k.

Hence P_k is invertible and, by padding the $(r+1)m$-th row of P_k^{-1} with zeros, we obtain the row $(\gamma_{k,j})_{j\in\mathbb{Z}}$ with the row index k of the left-inverse Γ.

Step 2. We show that this left inverse Γ defines a bounded operator on $\ell^2(\mathbb{Z})$.

By construction the j-th row $(\gamma_{k,j})$ of Γ has at most N non-zero entries between $k_1 = k-(r+1)m+1$ and k_1+N-1. To apply Lemma 15, we need to show that the entries of Γ are uniformly bounded, or equivalently, that the entries of P_k^{-1} are bounded with a bound that does not depend on k.

We set up a compactness argument similar to the proof of Theorem 8.

We begin with the simple observation that

$$g(\xi_j-\eta_l) = g\big((\xi_j-\eta_1)-(\eta_l-\eta_1)\big), \quad j,l = 1,\dots,N\,.$$

Let S be the N-dimensional simplex

$$S = \{\tau = (\tau_1,\dots,\tau_N)\in\mathbb{R}^N:\ 0\le\tau_1\le\dots\le\tau_N\le(N-1)Q_Y\}\,. \tag{27}$$

Although the finite sequences $(\xi_j)_{1\le j\le N}$ and $(\eta_l)_{1\le l\le N}$ depend on the row index k (and we should write $\xi_j^{(k)}$ and $\eta_l^{(k)}$ to make the dependence explicit), we always have

$$0<\xi_1-\eta_1<\xi_N-\eta_1<\eta_N-\eta_1\le(N-1)Q_Y\,.$$

Consequently,

$$(\xi_1-\eta_1,\dots,\xi_N-\eta_1)\in S, \quad\text{and}\quad (0,\eta_2-\eta_1,\dots,\eta_N-\eta_1)\in S\,.$$

Let $q := \min\{q_{X'},q_Y\}>0$ be the minimum of the separation distances of the quasi-uniform sets X' and Y and let

$$S_q = \{\tau=(\tau_1,\dots,\tau_N)\in S: \tau_{j+1}-\tau_j\ge q \text{ for } 1\le j\le N-1\}\,.$$

Then S_q is compact and

$$(\xi_1-\eta_1,\dots,\xi_N-\eta_1)\in S_q \quad\text{and}\quad (0,\eta_2-\eta_1,\dots,\eta_N-\eta_1)\in S_q\,.$$

Finally, we define the compact set

$$\begin{aligned} K = \{(\tau,\theta)\in S_q\times S_q:\ & \tau_j+\epsilon\le\theta_{j+m} \ \text{ for }\ 1\le j\le N-m,\\ & \theta_j+\epsilon\le\tau_{j+n} \ \text{ for }\ 1\le j\le N-n\ \}\,. \end{aligned}$$

The assumption $C_r(\epsilon)$ implies that

$$\Big((\xi_1-\eta_1,\dots,\xi_N-\eta_1),(0,\eta_2-\eta_1,\dots,\eta_N-\eta_1)\Big)\in K\,.$$

Clearly, the Schoenberg-Whitney conditions are satisfied for every point $(\tau,\theta)\in K$ and therefore every $N\times N$-matrix $(g(\tau_j-\theta_l))$ has positive determinant. Since the determinant depends continuously on (τ,θ) and K is compact, we conclude that

$$\inf_{(\tau,\theta)\in K}\det\Big(g(\tau_j-\theta_l)\Big)=\delta>0.$$

This construction implies that $\det P_k\ge\delta>0$ for every k. As in the proof of Theorem 8 we use Cramer's rule and conclude that all entries of P_k^{-1} are bounded by $(N-1)!\,\delta^{-1}\|g\|_\infty^{N-1}$.

The assumptions of the modified Schur test are satisfied, and Lemma 15 yields that the matrix Γ is bounded as an operator on $\ell^2(\mathbb{Z})$. Finally, Lemma 13 implies that X is a set of sampling for $V_Y(g)$. ■

Corollary 17. *Assume that g is totally positive of finite order $M \geq 2$ and $Y = h\mathbb{Z}$. Let $\alpha = \sup_{j\in\mathbb{Z}}(x_{j+1} - x_j)$ be the maximum gap between consecutive sampling points. If $\alpha < h$, then (x_j) is a set of sampling for $V_Y(g)$.*

Proof. The assumption of Theorem 16 is verified with $\epsilon = h - \alpha$. ■

REMARKS: 1. So far the conclusion of Corollary 17 was known only for B-splines as generators [2]. For other generators only qualitative results were known [1] or weak estimates far from the correct sampling density [1, 3, 34].

2. If $Y = h\mathbb{Z}$ and X satisfies condition $C_r(\epsilon)$, then the largest possible gap of consecutive points in X is $2h - 2\epsilon$. Of course, large gaps have to be compensated by a higher density of neighboring points so that the condition $|(y_k + \epsilon, y_{k+r} - \epsilon) \cap X'| \geq r + 1$ is still satisfied. By refining the compactness argument in the proof of Theorem 16, it is possible to derive even weaker conditions for X to be a set of sampling for a shift-invariant space $V_Y(g)$ with totally positive generator g.

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